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Truncated global transfer matrix: trigonometric Goryachev–Chaplygin gyrostat

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Abstract. The RTT relation with truncation is solved for the R-matrix associated with the six-vertex model. It turns out that the finite expansion of the global solution of the transfer matrix $T(u)$ in terms of the spectral parameter u can be realized by the defined algebra $\mathcal{A}^q$ formed by $sl_q(2)$ and the non-commutative coordinates. The Hamiltonian of the system has been found, which is the trigonometric deformation of the Goryachev–Chaplygin gyrostat.

1. Introduction

The RTT relation plays a central role in establishing the connection between quantum integrable systems and the quantum group symmetry [1–6]. For a given R-matrix satisfying the Yang–Baxter equation (YBE) a variety of transfer matrices T can be found to satisfy the RTT relation. When the R-matrix is rational it defines the Yangian [2] for some Lie algebras. For an R-matrix taking a trigonometric solution of the YBE, the usual q-affine quantum algebras can be defined [3–6]. For a given $R(u)$ matrix satisfying the YBE the commutation relations for each element of the transfer matrix $T(u)$ can be set up which form infinitely-dimensional algebras. The infinite expansion of the transfer matrix $T(u)$ in terms of the spectral parameter u leads to Yangian for rational $R(u)$ matrices [2–4]. The $Y(g)$ contains a set $\{I_a, J_a\}$ and its mapping where $\{I_a\}$ form a simple Lie algebra g and $\{J_a\}$ satisfy the relations given by Drinfeld [2]. This approach has further been proved to be related to the RTT approach [4].

On the other hand, it is well known that if $L_j(u)$, where j is the site index, satisfies the RLL relation, then so does $T(u) = \prod_{j=1}^{N} L_j(u)$, namely, $T(u)$ can be an N-polynomial in u if $L_j(u)$ is linear in u. However, as pointed out by Sklyanin [7], there exists a new type of solution of RTT, which cannot be proved to possess the above-mentioned form. An explicit example is the Goryachev–Chaplygin (G–C) top [8] in which $T(u)$ has the truncation form

$$T(u) = \sum_{n=0}^{3} T^{(n)} u^{-n}. \tag{1.1}$$

However, this is a 'global type' of solution of RTT, i.e. no corresponding local $L_j(u)$ is found [7]. Actually the basic knowledge on the long-ranged-interaction models had shown the validity of the statement. In the example of the G–C top the truncated $T(u)$ is a mapping of $E(3)$ formed by the angular momentum $\boldsymbol{L}$ and coordinates $\boldsymbol{r} = (x_1, x_2, x_3)$.

This extension is different from the $T(u)$ viewed as the mapping of the $Y(SL(2))$ formed by $\boldsymbol{I} = \boldsymbol{J}$ and $\boldsymbol{J}$ which satisfied the Drinfeld relations [2].

The truncated $T(u)$ possesses physical significance, including the G–C gyrostat as the simplest example. An interesting question arises: how can one extend the idea to the trigonometric case? For a given six-vertex form of $R(u)$ matrix we should find a truncated global trigonometric $T(u)$ which takes the G–C gyrostat as its rational limit. We call it the trigonometric extension of the G–C gyrostat.

This paper is organized as follows. In section 2 the general formulation of the truncated RTT with $R(u) = u + P$ is discussed. It will be emphasized that the truncated $T(u)$ is related to the condition

$$T^{(0)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{1.2}$$

In section 3 the general relations for the truncated trigonometric transfer matrix $T^{(n)}(u)(|n| \leqslant 3)$ will be shown. In section 4 a particular realization of $T^{(n)}(u)(|n| \leqslant 3)$ associated with the six-vertex form of the R-matrix is given. In section 5 we rewrite the solution in an appropriate form which takes the G–C gyrostat solution as its rational limit. Finally, we check the results by taking the rational limit.

2. Truncated $t(u)$ associated with $SL(2)$

For a given solution of the Yang–Baxter equation

$$\check{R}_{12}(u)\check{R}_{23}(u+v)\check{R}_{12}(v) = \check{R}_{23}(v)\check{R}_{12}(u+v)\check{R}_{23}(u) \tag{2.1}$$

the RTT relation

$$\check{R}(u-v)(t(u)\otimes t(v)) = (t(v)\otimes t(u))\check{R}(u-v) \tag{2.2}$$

where $\check{R}(u) = PR(u)$ is a rational solution of the YBE, leads to the quantum group through the defining relations for the generators $t(u) = (t(u))_{i,j=1}^{n}$ [2–4]. For a rational solution of the YBE, associated with $SL(2)$, one uses $t(u)$ to express the transfer matrix

$$t(u) = t^{(0)} + \sum_{n=1}^{\infty} u^{-n} t^{(n)} \tag{2.3}$$

where $t^{(n)}$ form a Yangian.

Let us start from the simplest rational solution of the YBE

$$\check{R}(u) = uP + I \tag{2.4}$$

where P is 4×4 representation of the permutation operator. Substituting (2.4) into (2.2) we obtain

$$[t_{ab}^{(0)}, t_{cd}^{(n)}] = 0 \qquad (a, b, c, d = 1, 2) \tag{2.5}$$

$$[t_{bc}^{(n+1)}, t_{ad}^{(m)}] - [t_{bc}^{(n)}, t_{ad}^{(m+1)}] + t_{ac}^{(n)} t_{bd}^{(m)} - t_{ac}^{(m)} t_{bd}^{(n)} = 0 \qquad (m, n \geqslant 0). \tag{2.6}$$

It is well known that the deformed determinant

$$\det t(u) = t_{11}(u)t_{22}(u-1) - t_{12}(u)t_{21}(u-1) \tag{2.7}$$

commutes with any elements of $t(v)$:

$$[\det t(u), t_{ab}(v)] = 0. \tag{2.8}$$

From (2.8) it follows that

$$[C_n, C_m] = 0 \qquad [C_n, t^{(n)}] = 0 \tag{2.9}$$

where

$$\det t(u) = \sum_{n=0}^{\infty} u^{-n} C_n. \tag{2.10}$$

It is easy to prove that $t^{(0)}$ takes the form

$$t^{(0)} = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} \tag{2.11}$$

where μ is a complex parameter. The standard choice is $\mu = 1$. However, if we take $\mu = 0$ the algebraic structure will be tremendously modified. Let us study the consequences.

First, the $t_{22}^{(1)}$ is a centre of the algebra. It is easy to obtain from (2.5) and (2.6) that

$$[t_{12}^{(n)}, t_{22}^{(1)}] = [t_{21}^{(n)}, t_{22}^{(1)}] = [t_{11}^{(n)}, t_{22}^{(1)}] = 0 \tag{2.12}$$

and

$$[t_{12}^{(1)}, t_{11}^{(n)}] = t_{12}^{(n)} \qquad [t_{21}^{(1)}, t_{11}^{(n)}] = -t_{21}^{(n)} \qquad [t_{12}^{(1)}, t_{21}^{(n)}] = t_{22}^{(n)} \tag{2.13}$$

where the relations with interchange of (n) and (1) are also valid. Obviously, (2.13) contains a Heisenberg algebra, for example, by setting $n = 1$ and $t_{22}^{(1)} = 0$:

$$t_{11}^{(1)} = \mathrm{i}p \qquad t_{12}^{(1)} = \mathrm{e}^{-q} \qquad t_{21}^{(1)} = \mathrm{e}^{+q} \tag{2.14}$$

if $[p, q] = -\mathrm{i}$. This is the reason why we call the algebra Heisenberg type.

By taking the product of the local solutions given by (2.14), the Toda lattice model is obtained. As we have emphasized, this is trivial for our discussion. What we are interested in is finding a global solution which cannot be decomposed into the local products. Equation (2.6) is equivalent to the following independent sets of relations:

$$[t_{22}^{(n)}, t_{11}^{(2)}] + t_{21}^{(n)} t_{12}^{(1)} - t_{21}^{(1)} t_{12}^{(n)} = 0 \tag{2.15}$$

$$\begin{aligned} t_{12}^{(n+1)} &= [t_{12}^{(2)}, t_{11}^{(n)}] + t_{11}^{(n)} t_{12}^{(1)} - t_{11}^{(1)} t_{12}^{(n)} \\ t_{21}^{(n+1)} &= [t_{11}^{(n)}, t_{21}^{(2)}] + t_{11}^{(n)} t_{21}^{(1)} - t_{11}^{(1)} t_{21}^{(n)} \\ t_{22}^{(n+1)} &= [t_{12}^{(2)}, t_{21}^{(n)}] + t_{11}^{(n)} t_{21}^{(1)} - t_{11}^{(1)} t_{22}^{(n)} \end{aligned} \tag{2.16}$$

for $n \geqslant 2$ and

$$[t_{ij}^{(n)}, t_{ij}^{(m)}] = 0 \qquad [t_{ij}^{(n)}, t_{kl}^{(m)}] = [t_{ij}^{(m)}, t_{kl}^{(n)}] = 0 \tag{2.17}$$

where $i, j, k, l = 1, 2$.

The Casimirs C_n for $\mu = 0$ in (2.11) are given by

$$\begin{aligned} C_0 &= 0 \qquad C_1 = t_{22}^{(1)} \\ C_j &= t_{22}^{(j)} + \sum_{\substack{m+l=j \\ m,l\neq 0}} \frac{(l+m-1)!}{(m-1)!l!} t_{22}^{(m)} \\ &\quad + \sum_{\substack{m+n+l=j \\ m,n\neq 0}} \frac{(l+m-1)!}{(m-1)!l!} (t_{11}^{(n)} t_{22}^{(m)} - t_{12}^{(n)} t_{21}^{(m)}) \qquad (j > 1) \end{aligned} \tag{2.18}$$

$t(u)$ is called truncated if $t^{(m)} = 0$ $(m \geqslant 4)$.

Obviously if $t^{(m)} = 0$, then $t_{ab}^{(m+1)} = 0$ by virtue of (2.16). To solve (2.13) and (2.15)–(2.17) with the truncation at $t_{ab}^{(4)} = 0$, we set the ansatz (motivated by Sklyanin [7])

$$t_{11}^{(1)} = \alpha p \qquad t_{22}^{(1)} = 0 \qquad t_{12}^{(1)} = \beta \mathrm{e}^{\tau q} x_{+} \qquad t_{21}^{(1)} = \gamma \mathrm{e}^{-\tau q} \tag{2.19}$$

and

$$t_{11}^{(2)} = f_1 J^2 + f_2 J_3^2 + f_3 p J_3 + f_4 x_- + f_5 \tag{2.20}$$

$$t_{11}^{(3)} = (p + g_3 J_3)(g_1 J^2 + g_2 J_3^2 + g_5) + g_4 [J_-, x_3]_+ \tag{2.21}$$

where $f_1, \ldots, f_5$, $g_1, \ldots, g_4$, α, β, γ and τ are parameters to be determined, $J_\pm = J_1 \pm \mathrm{i} J_2$ and $x_\pm = x_1 \pm \mathrm{i} x_2$ obey the commutation relations $(i, j, k = 1, 2, 3)$

$$[J_i, J_j] = -\mathrm{i}\epsilon_{ijk} J_k \tag{2.22}$$

$$[J_i, x_j] = -\mathrm{i}\epsilon_{ijk} x_k \tag{2.23}$$

$$[x_i, x_j] = [p, J_i] = [p, x_i] = [q, x_i] = [q, J_i] = 0 \tag{2.24}$$

$$[p, q] = -\mathrm{i} \tag{2.25}$$

and the relation

$$\sum_{i=1}^{3} J_i x_i = 0. \tag{2.26}$$

Substituting (2.19)–(2.26) into (2.15)–(2.17) we find

$$\begin{aligned} &\tau = -\mathrm{i}\alpha^{-1} \qquad f_1 = -\tfrac{1}{4} \qquad f_2 = -\tfrac{3}{4} \qquad f_3 = \alpha \qquad f_5 = -\tfrac{1}{16} \\ &g_1 = -g_2 = -\tfrac{1}{4}\alpha \qquad g_3 = -\alpha^{-1} \qquad g_4 = \tfrac{1}{4} f_4 \qquad g_5 = -\tfrac{1}{16}\alpha \end{aligned} \tag{2.27}$$

and the solution of the RTT relation for (2.4) at $\mu = 0$ in (2.11):

$$\begin{aligned} t_{11}^{(2)} &= -\tfrac{1}{4}(J^2 + 3J_3^2 + \tfrac{1}{4}) + \alpha p J_3 + f x_- \qquad t_{22}^{(2)} = \gamma\beta x_+ \\ t_{12}^{(2)} &= -\beta \mathrm{e}^{\tau q}(-\tfrac{1}{4}[J_+, x_3]_+ + x_+(J_3 - \alpha p)) \qquad t_{21}^{(2)} = \gamma \mathrm{e}^{-\tau q} J_3 \\ t_{11}^{(3)} &= -\tfrac{1}{4}\alpha(p - \alpha^{-1} J_3)(J^2 - J_3^2 + \tfrac{1}{4}) + \tfrac{1}{4} f [J_-, x_3]_+ \\ t_{22}^{(3)} &= \tfrac{1}{4}\beta\gamma[J_+, x_3]_+ \\ t_{12}^{(3)} &= -\beta \mathrm{e}^{\tau q}\{f x_3^2 - \tfrac{1}{4}\alpha[J_+, x_3]_+(p - \alpha^{-1} J_3)\} \\ t_{21}^{(3)} &= -\tfrac{1}{4}\gamma \mathrm{e}^{-\tau q}(J^2 - J_3^2 + \tfrac{1}{4}) \end{aligned} \tag{2.28}$$

and

$$\det t(u) = u^{-3}(u-1)^{-3} f (u^2 - u + \tfrac{3}{16}) \sum_{i=1}^{3} x_i^2. \tag{2.29}$$

It is easy to check that $\sum_{i=1}^{3} x_i^2$ commutes with $t_{ab}^{(n)} (n \leqslant 3)$. It follows that

$$\begin{aligned} \operatorname{tr} t(u) = 1 &+ u^{-1}\alpha p + u^{-2}\{-\tfrac{1}{4}(J^2 + 3J_3^2 + \tfrac{1}{4}) + \alpha p J_3 + f x_- + \beta\gamma x_+\} \\ &+ \tfrac{1}{4} u^{-3}\{-(\alpha p - J_3)(J^2 - J_3^2 + \tfrac{1}{4}) + f[J_-, x_3]_+ + \beta\gamma[J_+, x_3]_+\} \end{aligned} \tag{2.30}$$

where $[A, B]_+ = AB + BA$ and $f = f_4$.

The coefficient of u^{-2} can be viewed as the Hamiltonian of the system, and hence the truncated $t(u)$ given by (2.28) and (2.30) is interpreted as a Goryachev–Chaplygin gyrostat [7]. Actually, by making the rotation transformation about the x_3-axis for both coordinates and angular momentum (denote by $\boldsymbol{x}'$ and $\boldsymbol{J}'$, respectively) we obtain the conserved quantities [7]

$$H_p = \tfrac{1}{2}\{\tfrac{1}{4}\lambda(J'^2 + 3J_3'^2) - b x_1' - \alpha p J_3'\} + \tfrac{1}{16}\lambda \tag{2.31}$$

$$G_p = -\tfrac{1}{4}\{(\alpha p - 1) J_3'(J'^2 - J_3'^2 + \tfrac{1}{4}) + b[x_3, J_1']_+\} \tag{2.32}$$

where $b^2 = 4\beta\gamma f$.

Let us summarize the procedure for solving the truncated $t(u)$ as follows.

(1) Making the truncation at $n = 4$ in the expansion (2.3) and substituting it into the RTT relation (2.2), find the relations (2.12), (2.13) and (2.15)–(2.17).

(2) Find a realization for (2.12), (2.13) and (2.15)–(2.17), namely, use the simple algebra (2.23)–(2.26) to express $t_{ab}^{(n)}(n \leqslant 3)$ such that the RTT relation (2.2) is satisfied.

(3) Obtain the conserved quantities and set up the corresponding models.

Condition (2.26) and $\sum_i x_i^2 = r^2$ are nothing but the Casimirs of the algebra (2.22)–(2.24). It is well known that equation (2.26) leads to the vanishing monopole contribution associated with $E(3)$ topology, whereas (2.29) indicates that r^2 is the deformed determinant. It is natural to extend the above idea to the transfer matrix $T(u)$ associated with the trigonometric solution $\check{R}(x)(x = \mathrm{e}^{\mathrm{i}u})$ of the YBE. The process is parallel to the q-deformation of the rational case. First, we make $T(x)$ truncated and substitute it into the RTT relation for $R(x)$ being a trigonometric solution of the YBE. Next, we find an algebraic set which is the q-deformation of (2.23)–(2.26) such that the expressed $T(x)$ satisfies the RTT relation for the trigonometric case. Finally, such a q- deformation should include the results shown by (2.1)–(2.3) as a rational limit.

It is well known that the rational solution of $\check{R}(u)$ given by (2.4) can be regarded as the rational limit of the trigonometric R-matrix describing the six-vertex model [10]:

$$\check{R}_T(x) = \begin{bmatrix} a(x) & & & \\ & w & b(x) & \\ & b(x) & w & \\ & & & a(x) \end{bmatrix} \tag{2.33}$$

$$\check{R}_T(xy^{-1})(T(x) \otimes T(y)) = (T(y) \otimes T(x))\check{R}_T(xy^{-1}) \tag{2.34}$$

where $a(x) = qx - q^{-1}x^{-1}$, $b(x) = x - x^{-1}$, $w = q - q^{-1}$, $x = \mathrm{e}^u$ and $q = \mathrm{e}^\gamma$. Following Drinfeld [3] the transfer matrix $T(u)$ obeying (2.34) for $\check{R} = \check{R}_T$ should have the expansion form

$$T(x) = \sum_{n=-\infty}^{+\infty} x^n T^{(n)}. \tag{2.35}$$

We want to find the truncated solution $T^{(u)}(|n| \leqslant 3)$, i.e. that which possesses the form $T(x) = \sum_{n=-3}^{3} x^n T^{(n)}$ which takes the solution (2.28) as a rational limit. We shall see that this extension will give rise to non-commutative coordinates.

3. General formula for truncated $T^{(m)}(x)$ at $m = 4$

For the R-matrix given by (2.33) the RTT relation is shown by (2.34). Substituting

$$T(x) = \begin{bmatrix} T_{11}(x) & T_{12}(x) \\ T_{21}(x) & T_{22}(x) \end{bmatrix} = \sum_{n=-\infty}^{+\infty} x^n T^{(n)} \tag{3.1}$$

or

$$T_{ab}(x) = \sum_{n=-\infty}^{\infty} x^n T_{ab}^{(n)} (a, b = 1, 2) \tag{3.2}$$

into (2.33) we derive

$$[T_{ab}^{(k)}, T_{ab}^{(j)}] = 0 \qquad [T_{11}^{(k)}, T_{22}^{(j)}] = [T_{11}^{(j)}, T_{22}^{(k)}] \qquad [T_{12}^{(k)}, T_{21}^{(j)}] = [T_{12}^{(j)}, T_{21}^{(k)}]$$
$$qT_{aa}^{(k-1)}T_{ab}^{(j+1)} - q^{-1}T_{aa}^{(k+1)}T_{ab}^{(j+1)} = T_{ab}^{(j+1)}T_{aa}^{(k-1)} - T_{ab}^{(j-1)}T_{aa}^{(k+1)} + wT_{aa}^{(j)}T_{ab}^{(k)}$$
$$qT_{ab}^{(k-1)}T_{aa}^{(j+1)} - q^{-1}T_{ab}^{(k+1)}T_{aa}^{(j-1)} = T_{aa}^{(j+1)}T_{ab}^{(k-1)} - T_{aa}^{(j-1)}T_{ab}^{(k+1)} + wT_{ab}^{(j)}T_{aa}^{(k)}$$

$$qT_{ba}^{(j+1)}T_{aa}^{(k-1)} - q^{-1}T_{ba}^{(j-1)}T_{aa}^{(k+1)} = T_{aa}^{(k-1)}T_{ba}^{(j+1)} - T_{aa}^{(k+1)}T_{ba}^{(j-1)} + wT_{ba}^{(k)}T_{aa}^{(j)}$$
$$qT_{aa}^{(j+1)}T_{ba}^{(k-1)} - q^{-1}T_{aa}^{(j-1)}T_{ba}^{(k+1)} = T_{ba}^{(k-1)}T_{aa}^{(j+1)} - T_{ba}^{(k+1)}T_{aa}^{(j-1)} + wT_{aa}^{(j)}T_{ba}^{(k)}$$
$$[T_{22}^{(k-1)}, T_{11}^{(j+1)}] - [T_{22}^{(k+1)}, T_{11}^{(j-1)}] = w(T_{12}^{(j)}T_{21}^{(k)} - T_{12}^{(k)}T_{21}^{(j)})$$
$$[T_{21}^{(k-1)}, T_{12}^{(j+1)}] - [T_{21}^{(k+1)}, T_{12}^{(j-1)}] = w(T_{11}^{(j)}T_{22}^{(k)} - T_{11}^{(k)}T_{22}^{(j)}). \qquad (3.3)$$

The inverse of $T(x)$ is given by

$$[T(x)]^{-1} = [\det_q T(x)]^{-1}\begin{bmatrix} T_{22}(q^{-1}x) & -T_{12}(q^{-1}x) \\ -T_{21}(q^{-1}x) & T_{11}(q^{-1}x) \end{bmatrix} \qquad (3.4)$$

where

$$\det_q T(x) = T_{11}(x)T_{22}(q^{-1}x) - T_{12}(x)T_{21}(q^{-1}x). \qquad (3.5)$$

It is easy to show that

$$[\det_q T(x), T_{ab}(y)] = 0. \qquad (3.6)$$

Correspondingly

$$\det_q T(x) = \sum_{n=0} x^{-n}C_n \qquad C_n = \sum_{k+j=n} q^{-j}(T_{11}^{(k)}T_{22}^{(j)} - T_{12}^{(k)}T_{21}^{(j)}) \qquad (3.7)$$

and

$$\mathrm{tr}\, T(x) = \sum_n (T_{11}^{(n)} + T_{22}^{(n)})x^n. \qquad (3.8)$$

$T^{(n)}$ is called truncated if $T^{(n)} = 0$ for $|n| \geqslant 4$.

To solve (3.3) we take the ansatz (motivated by the rational correspondence)

$$T_{11}^{(\pm2)} = T_{11}^{(0)} = T_{22}^{(\pm3)} = T_{22}^{(\pm2)} = T_{22}^{(0)} = T_{12}^{(\pm3)} = T_{12}^{(\pm1)} = T_{21}^{(\pm3)} = T_{21}^{(\pm1)} = 0 \qquad (3.9)$$

which makes (3.3) reduce to the following set:

$$[T_{ab}^{(k)}, T_{ab}^{(j)}] = 0 \qquad q^{\pm1}T_{11}^{(\pm3)}T_{12}^{(j)} = T_{12}^{(j)}T_{11}^{(\pm3)}$$
$$q^{\mp1}T_{11}^{(\pm3)}T_{21}^{(j)} = T_{21}^{(j)}T_{11}^{(\pm3)}$$
$$q^{\pm1}T_{21}^{(\pm2)}T_{22}^{(j)} = T_{22}^{(j)}T_{21}^{(\pm2)} \qquad (j = \pm1, \mp1)$$
$$q^{\pm1}T_{22}^{(j)}T_{12}^{(\pm2)} = T_{12}^{(\pm2)}T_{22}^{(j)} \qquad (j = \pm1, \mp1)$$
$$q^{\pm1}T_{22}^{(\pm1)}T_{21}^{(0)} = T_{21}^{(0)}T_{22}^{(\pm1)} \pm wT_{22}^{(\mp1)}T_{21}^{\pm2}$$
$$q^{\pm1}T_{12}^{(0)}T_{22}^{(\pm1)} = T_{22}^{(\pm1)}T_{12}^{(0)} \pm wT_{12}^{(\pm2)}T_{22}^{(\mp1)}$$
$$q^{\pm1}T_{11}^{(j\pm1)}T_{21}^{(\pm2)} = T_{21}^{(\pm2)}T_{11}^{(j\pm1)} \pm wT_{11}^{(\pm3)}T_{21}^{(j)} \qquad (j = 0, \mp2)$$
$$q^{\pm1}T_{12}^{(\pm2)}T_{11}^{(j\pm1)} = T_{11}^{(j\pm1)}T_{12}^{(\pm2)} \pm wT_{12}^{(j)}T_{11}^{(\pm3)} \qquad (j = 0, \mp2)$$
$$q^{\pm1}T_{21}^{(0)}T_{11}^{(j\pm1)} \pm wT_{21}^{(\mp2)}T_{11}^{(\pm3)} = T_{11}^{(\pm1)}T_{21}^{(0)} \pm wT_{21}^{(\pm2)}T_{11}^{(\pm1)}$$
$$q^{\pm1}T_{11}^{(j\pm1)}T_{12}^{(0)} \pm wT_{11}^{(\pm3)}T_{12}^{(\mp2)} = T_{12}^{(0)}T_{11}^{(\pm1)} \pm wT_{11}^{(\mp1)}T_{12}^{(\pm2)}$$
$$[T_{21}^{(\pm2)}, T_{12}^{(\mp2)}] = 0 \qquad [T_{21}^{(\pm2)}, T_{12}^{(\pm2)}] = \mp wT_{11}^{(\pm3)}T_{22}^{(\pm1)}$$
$$[T_{21}^{(0)}, T_{12}^{(0)}] = \pm w(T_{11}^{(\mp1)}T_{22}^{(\pm1)} - T_{11}^{(\pm1)}T_{22}^{(\mp1)})$$
$$[T_{11}^{(\pm3)}, T_{22}^{(j)}] = 0 \qquad (j = \pm1, \mp1)$$
$$[T_{22}^{(\pm1)}, T_{11}^{(\pm1)}] = \pm w(T_{12}^{(0)}T_{21}^{(\pm2)} - T_{12}^{(\pm2)}T_{21}^{(0)})$$
$$[T_{22}^{(\pm1)}, T_{11}^{(\pm1)}] = \pm w(T_{12}^{(\mp2)}T_{21}^{(\pm2)} - T_{12}^{(\pm2)}T_{21}^{(\mp2)})$$
$$[T_{12}^{(\pm2)}, T_{21}^{(0)}] = [T_{12}^{(0)}, T_{21}^{(\pm2)}] = \pm wT_{11}^{(\pm3)}T_{22}^{(\mp1)} \qquad (3.10)$$

where, for example, $[T_{21}^{(\pm 2)}, T_{12}^{(\mp 2)}] = 0$ means that $[T_{21}^{(+2)}, T_{12}^{(-2)}] = [T_{21}^{(-2)}, T_{12}^{(+2)}] = 0$.

To solve the set of relations (3.10), which is still complicated, we set

$$\begin{aligned}
&T_{11}^{(\pm 3)} = \pm\lambda_3 \mathrm{e}^{\mp \mathrm{i}\eta P} \qquad T_{11}^{(\pm 1)} = A_\pm^{(0)} + \mathrm{e}^{\mp \mathrm{i}\eta P} A_\pm^{(1)} + \mathrm{e}^{\pm \mathrm{i}\eta P} A_\pm^{(2)} \\
&T_{12}^{(\pm 2)} = \mathrm{e}^{\mathrm{i}\xi Q \mp \mathrm{i}\eta P} E_{\pm 2} \qquad T_{12}^{(0)} = \mathrm{e}^{\mathrm{i}\xi Q}(E^{(0)} + \mathrm{e}^{-\mathrm{i}\eta P} E^{(-)} + \mathrm{e}^{\mathrm{i}\eta P} E^{(+)}) \\
&T_{21}^{(\pm 2)} = \alpha_2 K^{\pm 1} \mathrm{e}^{-\mathrm{i}\xi Q} \qquad T_{21}^{(0)} = \mathrm{e}^{-\mathrm{i}\xi Q} F_0 \qquad q = \mathrm{e}^{\mathrm{i}\xi\eta}
\end{aligned} \tag{3.11}$$

where λ_3, α_2, ξ and η are constants, P and Q satisfy

$$[P, Q] = -\mathrm{i} \tag{3.12}$$

and $A_\pm^{(i)} (i = 0, 1, 2)$, $E_{\pm 2}$, $E^{(j)} (j = 0, \pm)$, F_0 and K are operators commuting with P and Q. They will be determined by (3.10).

Proposition 3.1. With the ansatz (3.11) and by setting

$$K E_{\pm 2} = q^{-1} E_{\pm 2} K \tag{3.13}$$

the set of relations

$$\begin{aligned}
&A_\pm^{(2)} = \mp\lambda_3 K^{\pm 2} \qquad A_\pm^{(1)} = \pm\lambda_3 \alpha_2^{-1} K^{\mp} S \qquad F_0 = S \\
&E^{\pm} = -E_{\pm 2} K^{\pm 2} \qquad T_{22}^{(\pm 1)} = \pm\lambda_3^{-1} \alpha_2 E_{\pm 2} K^{\pm 1}
\end{aligned} \tag{3.14}$$

solves (3.10) if $A_\pm^{(0)}$, $E_{\pm 2}$, $E^{(0)}$ and S satisfy the following algebraic relations:

$$\begin{aligned}
&q^2 E_{-2} E_{+2} = E_{+2} E_{-2} \qquad [A_+^{(0)}, A_-^{(0)}] = [K, S] = 0 \\
&K A_{\pm -1}^{(0)} = q A_\pm^{(0)} K \qquad q^{\pm 1} E_{\pm 2} A_\mp^{(0)} = A_\mp^{(0)} E_{\pm 2} \\
&q^{\pm 1} E_{\pm 2} A_\pm^{(0)} - A_\pm^{(0)} E_{\pm 2} = \lambda_3 w E^{(0)} \\
&S A_\pm^{(0)} - q^{\mp 1} A_\pm^{(0)} S = \pm\alpha_2 w A_\mp^{(0)} K^{\pm 1} \\
&S E_{\pm 2} - q^{\pm 1} E_{\pm 2} S = \pm\alpha_2 w E_{\mp 2} K^{\mp 1}.
\end{aligned} \tag{3.15}$$

Proof. Substituting (3.11), (3.13) into (3.10) after calculations we find (3.14) under the condition (3.15). □

We will show that the realization of (3.13)–(3.15) can be made through the q-deformation of (2.22)–(2.26).

The set $\mathcal{A}^q = \{\hat{J}_\pm, \hat{J}_3; \hat{x}_\pm, \hat{x}_3\}$ is defined by the six generators indicated in the script parenthesis. They obey the commutation relations

$$\begin{aligned}
&[\hat{J}_\pm, \hat{J}_3] = \pm\hat{J}_3 \qquad [\hat{J}_+, \hat{J}_-] = g[\hat{J}_3]_q \\
&[\hat{x}_\pm, \hat{J}_3] = \pm\hat{x}_\pm \\
&[\hat{J}_\pm, \hat{x}_3] = a_\pm^{(1)} \hat{J}_\pm \hat{x}_3 + a_\pm^{(2)} \hat{x}_\pm \\
&[\hat{J}_\pm, \hat{x}_\mp] = b_\pm^{(1)} \hat{J}_\pm \hat{x}_\mp + b_\pm^{(2)} \hat{x}_3 \\
&\hat{x}_\mp \hat{x}_3 = f_\pm \hat{x}_3 \hat{x}_\pm \qquad [\hat{J}_\pm, \hat{x}_\pm] = [\hat{J}_3, \hat{x}_3] = 0
\end{aligned} \tag{3.16}$$

where $f_\pm$ and $a_\pm^{(1)}$ depend on q only and

$$[\hat{J}_3]_q = (q - q^{-1})^{-1} (q^{\hat{J}_3} - q^{-\hat{J}_3}). \tag{3.17}$$

g is arbitrary and will be restricted to be 2 or $(q + q^{-1})$ by the fact that the rational limit of (3.16) should be (2.22)–(2.25).

The sufficient choice

$$f_{\pm} = q^{\delta_{\pm}} \qquad a_{\pm}^{(2)} = \pm\tau_{\pm}K^{\pm\delta_{\pm}} \qquad b_{\pm}^{(2)} = \mp\tau_{\mp}^{-1}gK^{\mp\delta_{\pm}} \tag{3.18}$$

makes (3.16) self-consistent, where

$$K = \mathrm{e}^{\mathrm{i}\xi\eta\hat{J}_3} = q^{\hat{J}_3}. \tag{3.19}$$

$\delta_{\pm}$ are independent of q and obey $\delta_{+}+\delta_{-} = +1$ or -1, whereas $\tau_{\pm}$ may be either dependent on q or independent of q and $\lim_{q\to 1}\tau_{\pm} = 1$. Obviously, (3.16) looks like a deformation of $E(3)$ algebra [10].

Proposition 3.2. The set $\mathcal{A}^q$

$$[\hat{J}_{\pm}, \hat{x}_{\pm}] = [\hat{J}_3, \hat{x}_3] = 0 \tag{3.20a}$$

$$[\hat{J}_{\pm}, \hat{J}_3] = \pm\hat{J}_3, [\hat{J}_{+}, \hat{J}_{-}] = g[\hat{J}_3]_q \tag{3.20b}$$

$$[\hat{x}_{\pm}, \hat{J}_3] = \pm\hat{x}_{\pm} \tag{3.20c}$$

$$q^{\delta_{\mp}}\hat{J}_{\pm}\hat{x}_3 = \hat{x}_3\hat{J}_{\pm} \pm \tau_{\pm}K^{\pm\delta_{\pm}}\hat{x}_{\pm} \tag{3.20d}$$

$$[\hat{x}_{+}, \hat{x}_{-}] = 0 \qquad \hat{x}_{\pm}\hat{x}_3 = q^{\delta_{\pm}}\hat{x}_3\hat{x}_{\pm} \tag{3.20e}$$

$$q^{-1}\hat{J}_{\pm}\hat{x}_{\mp} = \hat{x}_{\mp}\hat{J}_{\pm} \mp \tau_{\mp}^{-1}qK^{\mp\delta_{\pm}}\hat{x}_3 \tag{3.20f}$$

forms the q-associative algebra $\mathcal{A}_{+}^{q}$ for $\delta_{+}+\delta_{-} = 1$ and $\mathcal{A}_{-}^{q}$ for $\delta_{+}+\delta_{-} = -1$. (3.20*a*)–(3.20*f*) are nothing but the q-deformation of (2.22)–(2.26).

Theorem 3.1. $\mathcal{A}_{+}^{q}$ and $\mathcal{A}_{-}^{q}$ satisfy the same relation

$$g[\hat{J}_3]_q\hat{x}_3 + \tau_{-}K^{-\delta_{+}}\hat{J}_{+}\hat{x}_{-} + \tau_{+}K^{\delta_{+}}\hat{J}_{-}\hat{x}_{+} = 0. \tag{3.21}$$

It can be directly checked, see the appendix.

To show the meaning of (3.21) let $q \to 1$ then $\mathcal{A}^q \to \mathcal{A}\{J_{\pm}, J_3, x_{\pm}, x_3\}$, so that (3.21) is reduced to (2.26). (3.21) means that the q-casimir has been chosen to be zero that corresponding to the vanishing monopole after q-deformation.

In the next section we shall prove that (3.20*a*)–(3.20*f*) (with the condition (3.21)) will solve (3.14) and (3.15), namely, they provide a realization of (3.10).

4. Realization of truncated $T^{(n)}(|n| \leqslant 3)$ in terms of $\mathcal{A}^q$

Theorem 4.1. $\mathcal{A}_{\pm}^{q}$ realize the trigonometric truncated $T^{(n)}$ ($T^{(n)} = 0$, for $|n| \geqslant 4$).

Proof. Setting

$$A_{\pm}^{(0)} = \lambda_{\pm}^{(1)}\hat{J}_{-}\hat{x}_3 + \lambda_{\pm}^{(2)}\hat{x}_{-}$$

$$E_{\pm 2} = \beta_{\pm}^{(1)}\hat{J}_{+}\hat{x}_3 + \beta_{\pm}^{(2)}\hat{x}_{+}$$

$$S = \lambda_1\hat{J}_{+}\hat{J}_{-} + \lambda_2$$

$$K = \exp(\mathrm{i}\xi\eta\hat{J}_3) \tag{4.1}$$

where $\lambda_{\pm}^{(i)}$, $\beta_{\pm}^{(i)}$ and λ_i $(i = 1, 2)$ are K-dependent operators. Substituting (4.1) into (3.13)–(3.15) and noting that $KA_{\pm}^{(0)}K^{-1} = qA_{\pm}^{(0)}$, $[K, S] = 0$, the constraints on the unknown parameters can completely be determined:

$$\alpha_2^2 = g\alpha^2 \qquad \lambda_1 = (g-1)wg^{-1}\alpha \qquad \lambda_2 = \alpha(q - K + K^{-1}) \tag{4.2}$$

and

$$\lambda_-^{(1)} = 0 \qquad \lambda_+^{(1)} = \lambda^{(+)} K^{2(1-\delta_+)-\beta}$$
$$\lambda_+^{(2)} = (1-q^{-1})^{-1} q^{-\delta_+} \tau_- K^{-1+\delta_+} \lambda_+^{(1)}$$
$$\lambda_-^{(2)} = (1-q^{-1})^{-1} q^{-1-\delta_+} \alpha^{-1}\alpha_2 \tau_- K^{-1+\delta_+} \lambda_+^{(1)} \qquad \beta_+^{(1)} = 0$$
$$\beta_-^{(2)} = -(1-q^{-1})^{-1} q^{-1+\delta_+} \tau_+ K^{\delta_+} \beta_-^{(1)} \qquad \beta_-^{(1)} = \beta^{(-)} K^{\beta}$$
$$\beta_+^{(2)} = (1-q^{-1})^{-1} q^{-4+\delta_+} \alpha^{-1}\alpha_2 \tau_+ K^{\delta_+-2} \beta_-^{(1)}$$
$$E^{(0)} = -\lambda_3^{-1} w^{-1} (1-q^{-1})^{-1} q^{-3+\delta_+} g\alpha^{-1}\alpha_2 K^{-1} \beta_-^{(1)} \lambda_+^{(1)}(qK)(\hat{x}_3)^2 \tag{4.3}$$

for $\delta_+ + \delta_- = 1$;

$$\lambda_+^{(1)} = 0, \lambda_{(-)}^{(1)} = \lambda^- K^{2(1+\delta_+)-\beta} \qquad \lambda_-^{(2)} = (1-q^{-1})^{-1} q^{-\delta_+} \tau_- K^{1+\delta_+} \lambda_-^{(1)}$$
$$\lambda_+^{(2)} = (1-q^{-1})^{-1} q^{-\delta_+} \alpha^{-1}\alpha_2 \tau_- K^{1+\delta_+} \lambda_-^{(1)} \qquad \beta_-^{(1)} = 0$$
$$\beta_+^{(2)} = -(1-q^{-1})^{-1} q^{1+\delta_+} \tau_+ K^{\delta_+} \beta_+^{(1)} \qquad \beta_+^{(1)} = \beta^{(+)} K^{\beta}$$
$$\beta_-^{(2)} = (1-q^{-1})^{-1} q^{4+\delta_+} \alpha\alpha_2^{-1} \tau_+ K^{\delta_++2} \beta_+^{(1)}$$
$$E^{(0)} = -\lambda_3^{-1} w^{-1} (1-q^{-1})^{-1} q^{3+\delta_+} g\alpha\alpha_2^{-1} K \beta_+^{(1)} \lambda_-^{(1)}(qK)(\hat{x}_3)^2 \tag{4.4}$$

for $\delta_+ + \delta_- = -1$, where δ_+, α, $\beta^{(\pm)}$, $\lambda^{(\pm)}$ and $\tau_\pm$ are arbitrary constants. Note that the parameters $\beta_\mp^{(1)}(K)$ depend on K and $\lambda_\pm^{(1)}(qK)$ on qK, respectively, whereas δ_+, α, $\beta^{(\pm)}$, $\lambda^{(\pm)}$ and $\tau_\pm$ are arbitrary constants. Thus (4.1)–(4.4) solve (3.13)–(3.15), namely they solve (3.10) which solves the truncated transfer matrix ($|m| \leqslant 3$) with special form (3.9). In other words, we have made the realization of $T^{(n)}(|n| \leqslant 3)$ with the help of q-deformed algebra $\mathcal{A}_\pm^q$. □

Theorem 4.2. The $\det_q T(x)$ for $T^{(n)}(x) = 0$ ($|n| \geqslant 4$) can be expressed by $\hat{J}_\pm$, $\hat{x}_\pm$ and $\hat{x}_3$ through

$$\det_q T(x) = \{x^2 + q^2 x^{-2} + \alpha^{-1}\alpha_2(1+q)\} C_{+2} \qquad (\alpha^{-1}\alpha_2 = \pm q^{\frac{1}{2}}) \tag{4.5}$$

where

$$C_{+2} = F_2 \left\{ \tau_- K^{\delta_+} \left(\hat{J}_+ \hat{J}_- \hat{x}_3 - \frac{\tau_+ K^{\delta_+} \hat{x}_+ \hat{x}_-}{1-q} + w^{-1} g(\hat{x}_3)^2 \right) \right\} \tag{4.6}$$

with

$$F_2 = \lambda_3^{-1} \alpha (1-q)^{-1} q^{1-\beta-\delta_+} \beta^{(+)} \lambda^{(-)} K^{2\delta_+} \tag{4.7}$$

that leads to, for $\delta_+ + \delta_- = -1$,

$$C_+ = F_1 \left\{ \tau_+ K^{\delta_+-1} \left(\hat{J}_- \hat{J}_+ \hat{x}_3 - \frac{\tau_- K^{\delta_+-1} \hat{x}_+ \hat{x}_-}{1-q^{-1}} + wg(\hat{x}_3)^2 \right) \right\} \tag{4.8}$$

where

$$F_1 = \lambda_3^{-1} \alpha (1-q^{-1})^{-1} q^{-2-\beta-\delta_+} \beta^{(-)} \lambda^{(+)} K^{2(1-\delta_+)}. \tag{4.9}$$

Proof. The non-vanishing C_n in (3.7) are

$$C_{\pm 4} = q^{\mp 1}(T_{11}^{(\pm 3)}T_{22}^{(\pm 1)} - q^{\mp 1}T_{12}^{(\pm 2)}T_{21}^{(\pm 2)})$$
$$C_{\pm 2} = q^{\pm 1}T_{11}^{(\pm 3)}T_{22}^{(\mp 1)} + q^{\mp 1}T_{11}^{(\pm 1)}T_{22}^{(\pm 1)} - T_{12}^{(\pm 2)}T_{21}^{(0)} - q^{\mp 2}T_{12}^{(0)}T_{21}^{(\pm 2)}$$
$$C_0 = qT_{11}^{(+1)}T_{22}^{(-1)} + q^{-1}T_{11}^{(-1)}T_{22}^{(+1)} - q^2 T_{12}^{(2)}T_{21}^{(-2)} - q^{-2}T_{12}^{(-2)}T_{21}^{(2)}T_{12}^{(0)}T_{21}^{(0)}. \tag{4.10}$$

Substituting (3.11)–(3.15) into $\det_q T(x)$ after calculations we obtain $C_{-2} = q^2 C_{+2}$, $C_0 = \alpha^{-1}\alpha_2(1+q)C_{+2}$, $C_{\pm 4} = 0$ and $C_{+2} = q^{-1}\alpha_2(\lambda_3^{-1}A_+^{(0)}E_{+2} - q^{-1}E^{(0)})K$ that lead to (4.5) and (4.6).

It is straightforward to obtain

$$\begin{aligned}\operatorname{tr} T(x) &= \lambda_3(\mathrm{e}^{-\mathrm{i}\eta P}x^3 - \mathrm{e}^{\mathrm{i}\eta P}x^{-3}) + (A_+^{(0)} + \mathrm{e}^{-\mathrm{i}\eta P}A_+^{(1)} + \mathrm{e}^{\mathrm{i}\eta P}A_+^{(2)} + \lambda_3^{-1}\alpha_2 E_{+2}K)x \\ &\quad + (A_-^{(0)} + \mathrm{e}^{-\mathrm{i}\eta P}A_-^{(2)} + \mathrm{e}^{\mathrm{i}\eta P}A_-^{(1)} - \lambda_3^{-1}\alpha_2 E_{-2}K^{-1})x^{-1}\end{aligned} \tag{4.11}$$

where $A_\pm^{(0)}$ and $E_{\pm 2}$ are given by (4.1), whereas $A_\pm^{(1)}$ and $A_\pm^{(2)}$ are given by (3.14).

From (4.11) it follows that besides P being conserved there are other two conserved quantities, $T_{11}^{(1)} + T_{22}^{(1)}$ and $T_{11}^{(-1)} + T_{22}^{(-1)}$. However, the form (3.8) is not good for taking the rational limit in comparison with (2.28). We shall present the equivalent form of $T^{(n)}(|n| \leqslant 3)$ to make further discussion more explicit.

5. The equivalent form of $T(x)$

Rewriting (3.1) in the form

$$\begin{aligned}T(x) = \sum_{n=-3}^{3} x^n T^{(n)} = T(u) &= \sum_{n,m} \sin^n u \cos^m u T^{(n,m)} \\ &= T^{(0,0)} + \sin u T^{(1,0)} + \cos u T^{(0,1)} + \sin^2 u T^{(2,0)} \\ &\quad + \sin u \cos u T^{(1,1)} + \sin^2 u \cos u T^{(2,1)} + \sin^3 u T^{(3,0)}\end{aligned} \tag{5.1}$$

where $x = \mathrm{e}^{\mathrm{i}u}$ and substituting (3.11)–(3.15), (4.1), (4.3) into (5.1) we find

$$T_{11}^{(3,0)} = -\mathrm{i}8\lambda_3\cos(\eta P) \qquad T_{11}^{(2,1)} = \mathrm{i}8\lambda_3\sin(\eta P)$$
$$T_{21}^{(2,0)} = -\mathrm{e}^{-\mathrm{i}\xi Q}4\alpha_2\cos(\xi\eta\hat{J}_3) \qquad T_{21}^{(1,1)} = -\mathrm{e}^{-\mathrm{i}\xi Q}4\alpha_2\sin(\xi\eta\hat{J}_3)$$

and

$$\begin{aligned}
T_{12}^{(2,0)} &= -2\mathrm{e}^{\mathrm{i}\xi Q}\beta_{\mp}^{(1)}\{\mathrm{e}^{\mathrm{i}\eta P}\hat{J}_+\hat{x}_3 \\
&\quad -(1-q^{-1})^{-1}q^{-2+\delta_+}\tau_+K^{\delta_+-1}\hat{x}_+(\mathrm{e}^{\mathrm{i}\eta P}K - q^{-1}\alpha^{-1}\alpha_2\mathrm{e}^{-\mathrm{i}\eta P}K^{-1})\} \\
T_{12}^{(1,1)} &= 2\mathrm{i}\mathrm{e}^{\mathrm{i}\xi Q}\beta_-^{(1)}\{-\mathrm{e}^{\mathrm{i}\eta P}\hat{J}_+\hat{x}_3 \\
&\quad +(1-q^{-1})q^{-2+\delta_+}\tau_+K^{\delta_+-1}\hat{x}_+(\mathrm{e}^{\mathrm{i}\eta P}K + q^{-1}\alpha^{-1}\alpha_2\mathrm{e}^{-\mathrm{i}\eta P}K^{-1})\} \\
T_{12}^{(0,0)} &= \mathrm{e}^{\mathrm{i}\xi Q}\beta_-^{(1)}\{-\lambda_3^{-1}w^{-1}(1-q^{-1})^{-1}q^{-3+\delta_+}g\alpha^{-1}\alpha_2K^{-1}\lambda_+^{(1)}(qK)(\hat{x}_3)^3 \\
&\quad +2\mathrm{i}(\hat{J}_+\hat{x}_3K^{-1} - (1-q^{-1})^{-1}q^{-2+\delta_+}\tau_+K^{\delta_+-1}\hat{x}_+(1+q^{-1}\alpha^{-1}\alpha_2) \\
&\quad \times\sin[\eta(P+\xi\hat{J}_3)])\} \\
T_{21}^{(0,0)} &= \mathrm{e}^{-\mathrm{i}\xi Q}\alpha\{(q-1)wg^{-1}\hat{J}_+\hat{J}_- + (q+\alpha^{-1}\alpha_2)(K + (1+\alpha^{-1}\alpha_2)K^{-1})\} \qquad (5.2) \\
T_{11}^{(1,0)} &= \mathrm{i}\lambda_+^{(1)}\{\hat{J}_-\hat{x}_3 + (1-q^{-1})^{-1}q^{-\delta_+}\tau_-K^{\delta_+-1}(1\mp q^{-\frac{1}{2}})\hat{x}_-\} \\
&\quad +2\mathrm{i}\lambda_3\{\pm q^{-\frac{1}{2}}(q-1)wg^{-1}\cos[\eta(P+\xi\hat{J}_3)]\hat{J}_+\hat{J}_- \\
&\quad +2\sin(\xi\eta\hat{J}_3)(\sin[\eta(P+\eta\hat{J}_3)]\mp\sin[\eta(P+\xi\hat{J}_3+\xi/2)]) \\
&\quad +\cos[\eta P(\pm q^{-\frac{1}{2}}(q+1)+2)]\}
\end{aligned}$$

$$T_{11}^{(0,1)} = \lambda_+^{(1)}\{\hat{J}_-\hat{x}_3 + (1-q^{-1})q^{-\delta_+}\tau_- K^{\delta_+-1}(1 \pm q^{-\frac{1}{2}})\hat{x}_-\}$$
$$-2\mathrm{i}\lambda_3\{\pm q^{-\frac{1}{2}}(q-1)wg^{-1}\hat{J}_+\hat{J}_- + 2(\pm\cos[\eta\xi(\hat{J}_3+\tfrac{1}{2})]$$
$$+\cos(\xi\eta\hat{J}_3))\}\sin[\eta(P+\xi\hat{J}_3)]$$
$$T_{22}^{(1,0)} = \mathrm{i}\lambda_3^{-1}\alpha_2\beta_-^{(1)}\{\hat{J}_+\hat{x}_3K^{-1} + (1-q^{-1})^{-1}q^{-2+\delta_+}\tau_+K^{\delta_+-1}\hat{x}_+(\pm q^{-\frac{1}{2}}-1)\}$$
$$T_{22}^{(0,1)} = \lambda_3^{-1}\alpha_2\beta_-^{(1)}\{-\hat{J}_+\hat{x}_3K^{-1} + (1-q^{-1})q^{-2+\delta_+}\tau_+K^{\delta_+-1}\hat{x}_+(\pm q^{-\frac{1}{2}}+1)\}$$

for $\delta_+ + \delta_- = 1$ and the similar expressions for $\delta_+ + \delta_- = -1$.

The $\det_q T(u)$ and $\mathrm{tr}\, T(u)$ are rewritten in the forms

$$\det_q T(u) = \{2q\cos(2u-\xi\eta) + \alpha^{-1}\alpha_2(1+q)\}C_{+2} \tag{5.3}$$

and

$$\mathrm{tr}\, T(u) = T_{11}^{(3,0)}\sin^3 u + T_{11}^{(2,1)}\sin^2 u\cos u + (T_{11}^{(1,0)} + T_{22}^{(1,0)})\sin u$$
$$+(T_{11}^{(0,1)} + T_{22}^{(0,1)})\cos u. \tag{5.4}$$

Following the inverse scattering methods, $T_{11}^{(3,0)}$ and $T_{11}^{(2,1)}$ correspond to the momentum conservation. The Hamiltonian is defined by

$$H = f_1\{T_{11}^{(1,0)} + T_{22}^{(1,0)}\} \tag{5.5}$$

and another constant of motion is

$$G = f_2\{T_{11}^{(0,1)} + T_{22}^{(0,1)}\} \tag{5.6}$$

where f_1 and f_2 are arbitrary constants.

The Hamiltonian of the system given by (5.1), (3.9), (2.33) and (2.34) can take the form

$$H = f_1(2\mathrm{i}\lambda_3\{\alpha\alpha_2^{-1}(q-1)wg^{-1}\cos[\eta(P+\xi\hat{J}_3)]\hat{J}_+\hat{J}_- + 2\sin(\xi\eta\hat{J}_3)$$
$$\times(\sin[\eta(P+\xi\hat{J}_3] - \alpha^{-1}\alpha_2 q^{\frac{1}{2}}\sin[\eta(P+\xi\hat{J}_3+\tfrac{1}{2})]) + (\alpha\alpha_2^{-1}(q+1)+2)$$
$$\times\cos(\xi P)\} + D_+) \tag{5.7}$$
$$G = f_2(-2\mathrm{i}\lambda_3\{\alpha\alpha_2^{-1}(q-1)wg^{-1}\hat{J}_+\hat{J}_- + 2(\alpha\alpha_2^{-1}q^{\frac{1}{2}}\cos[\xi\eta(\hat{J}_3+\tfrac{1}{2})])$$
$$+\cos(\xi\eta\hat{J}_3)\}\sin[\eta(P+\xi\hat{J}_3] + D_-) \tag{5.8}$$

where $D_\pm$ are given by

$$D_\pm = \epsilon_\pm\{\lambda_3^{-1}\alpha_2\beta_+^{(1)}(\pm\hat{J}_\pm\hat{x}_3K^{-1} + (1-q^{-1})^{-1}q^{-2+\delta_+}\tau_+K^{\delta_+-1}(\alpha^{-1}\alpha_2 q^{-1}$$
$$\mp 1)\hat{x}_+) + \lambda_+^{(1)}(\hat{J}_-\hat{x}_3 + (1-q^{-1})^{-1}q^{-\delta_+}\tau_-K^{1+\delta_+}(1\mp q^{-1}\alpha^{-1}\alpha_2)\hat{x}_-)\} \tag{5.9}$$

for $\delta_+ + \delta_- = 1$, and

$$D_\pm = \epsilon_\pm\{\lambda_3^{-1}\alpha_2\beta_+^{(1)}(\hat{J}_\pm\hat{x}_3K + (1-q)^{-1}q^{2+\delta_+}\tau_+K^{\delta_++1}(\alpha\alpha_2^{-1}q\mp 1)\hat{x}_+)$$
$$+\lambda_-^{(1)}(\mp\hat{J}_+\hat{x}_3 + (1-q)^{-1}q^{-\delta_+}\tau_-K^{1+\delta_+}(\alpha\alpha_2^{-1}q\mp 1)\hat{x}_-)\} \tag{5.10}$$

for $\delta_+ + \delta_- = -1$, with $\epsilon_+ = \mathrm{i}$, $\epsilon_- = 1$.

It is noted that there are two possibilities

$$\alpha = \pm q^{-\frac{1}{2}}\alpha_2 \tag{5.11}$$

in (5.9) and (5.10).

We would like to remark that for the given standard six-vertex R-matrix (2.33) we find a set of solutions for the truncated RTT relation (2.34). The solutions can be realized through the algebra (3.20*a*)–(3.20*f*) and corresponding conserved quantities (5.4) and (5.5). Equations (3.20*a*)–(3.20*f*) naturally yield the non-commutative geometry, since the considered system is axially symmetric so that the coordinates on the (x_1, x_2) plane commute with each other, whereas the third coordinates x_3 does not commute with them.

6. Rational limit

Let us first take $\alpha = -q^{-\frac{1}{2}}\alpha_2$. When $q \to 1$, i.e. $\eta \to 0$ by taking the arbitrariness of the parameters λ_3, α_2, $\beta^{(\pm)}$ and $\lambda^{(\pm)}$ into account, we assume

$$\begin{aligned} &\lambda_3 \to \mathrm{i}\tfrac{1}{8}\eta^{-3} \qquad \alpha_2 \to -\tfrac{1}{4}\gamma\eta^{-2} \qquad \beta^{-1} \to \mathrm{i}\tfrac{1}{4}\beta_-\eta^{-1} \\ &\lambda^{(+)} \to \tfrac{1}{2}f_+ \qquad \beta^{(+)} \to -\mathrm{i}\tfrac{1}{4}\beta_+\eta^{-1} \qquad \lambda^{(-)} \to \tfrac{1}{2}f_- \end{aligned} \tag{6.1}$$

as $q \to 1$, where λ, γ, $\beta_\pm$ and $f_\pm$ are q-independent parameters. For simplicity we take $\xi = 1$ and from (5.2) it follows that as $q \to 1$

$$\begin{aligned} &T_{11}^{(3,0)} \to \lambda\eta^{-3} \qquad T_{11}^{(2,1)} \to -\lambda\eta^{-2}P \\ &T_{21}^{(2,0)} \to \gamma\eta^{-2}\,\mathrm{e}^{-\mathrm{i}\xi Q} \qquad T_{21}^{(1,1)} \to \gamma\eta^{-1}\hat{J}_3\,\mathrm{e}^{-\mathrm{i}\xi Q}. \end{aligned} \tag{6.2}$$

Correspondingly we obtain for $\alpha = -q^{-\frac{1}{2}}\alpha_2(\lambda = 1)$

$$\begin{aligned} &T_{12}^{(2,0)} \to \eta^{-2}\beta_\mp\,\mathrm{e}^{\mathrm{i}Q}x_+ \\ &T_{12}^{(1,1)} \to -\eta^{-1}\beta_\mp\,\mathrm{e}^{\mathrm{i}Q}\{-\tfrac{1}{4}[J_+, x_3]_+ + x_+(P + J_3)\} \\ &T_{12}^{(0,0)} \to -\beta_\mp\,\mathrm{e}^{\mathrm{i}Q}\{f_\pm x_3^2 + \tfrac{1}{4}[J_+, x_3]_+(P + J_3)\} \\ &T_{21}^{(0,0)} \to -\tfrac{1}{4}\gamma\,\mathrm{e}^{-\mathrm{i}Q}(J^2 - J_3^2 + \tfrac{1}{4}) \\ &T_{11}^{(1,0)} \to \eta^{-1}\{-\tfrac{1}{4}(J^2 + 3J_3^2 + \tfrac{1}{4}) - \lambda PJ_3 + f_\pm x_-\} \\ &T_{11}^{(0,1)} \to \tfrac{1}{4}(P + J_3)(J^2 - J_3^2 + \tfrac{1}{4}) + \tfrac{1}{4}f_\mp[J_-, x_3]_+ \\ &T_{22}^{(1,0)} \to \gamma\beta_\mp\eta^{-1}x_+ \\ &T_{22}^{(0,1)} \to \tfrac{1}{4}\gamma\beta_\mp[J_+, x_3]_+ \end{aligned} \tag{6.3}$$

where the upper subindices correspond to $\delta_+ + \delta_- = 1$ and the lower ones to $\delta_+ + \delta_- = -1$. Taking $q \to 1$ and making the transformations

$$P \to -\lambda^{-1}\alpha' P \qquad Q \to -\lambda^{-1}\alpha' Q \qquad u \to \eta u \tag{6.4}$$

for $\delta_+ + \delta_- = 1$ we derive $T_{ab}^{(m,n)} \to t_{ab}^{(m,n)}$:

$$\begin{aligned} t_{11}(u) &= u^3 + \alpha' Pu^2 - \{-\tfrac{1}{4}(J^2 + 3J_3^2 + \tfrac{1}{4}) + \alpha' P + fx_-\}u \\ &\quad -\tfrac{1}{4}\alpha'(P - \alpha'^{-1}J_3)(J^2 - J_3^2 + \tfrac{1}{4}) + \tfrac{1}{4}f[J_-, x_3]_+ \\ t_{12}(u) &= \beta'\,\mathrm{e}^{-\mathrm{i}\alpha'^{-1}Q}\{u^2 + (\tfrac{1}{4}[J_+, x_3]_+ + x_+(\alpha P - \lambda J_3)u \\ &\quad +\tfrac{1}{4}\lambda^{-1}\alpha'[J_+, x_3]_+(P - \lambda\alpha'^{-1}J_3) - \lambda^{-1}fx_3^2\} \\ t_{21}(u) &= \gamma\,\mathrm{e}^{\mathrm{i}\alpha'^{-1}Q}\{u^2 + J_3 u - \tfrac{1}{4}(J^2 - J_3^2 + \tfrac{1}{4})\} \\ t_{22}(u) &= \gamma\beta'(x_+ u + \tfrac{1}{4}[J_+, x_3]_+) \end{aligned} \tag{6.5}$$

where $f = f_\pm$, $\beta' = \beta_\pm$. If we take

$$u \to -\tfrac{1}{2}u \qquad \alpha' = -\tfrac{1}{2} \qquad f = \tfrac{1}{4}b \qquad \beta' = -\tfrac{1}{2}b \qquad \gamma = -\tfrac{1}{2} \tag{6.6}$$

(6.5) is nothing but the result given by (2.28) (see Sklyanin [7]).

It can be proved that when $\delta_+ + \delta_- = -1$ we obtain the same rational limit. Besides, (5.3) limits to $(q \to 1)$

$$\det_q T(u) \underset{u\to\eta u}{\overset{q\to 1}{\to}} \gamma\beta' f(u - \tfrac{1}{4})(u - \tfrac{3}{4})(x_+x_- + x_3^2). \tag{6.7}$$

It is remarked that if the η-dependence of λ_3, α_2, $\beta^{(\pm)}$ and $\lambda^{(\pm)}$ in (6.1) is given in a different way it will give rise to a different rational limit. For instance, if we take $\xi = 1$ and let

$$\lambda_3 \to -\mathrm{i}\tfrac{1}{8}\lambda\eta^{-1} \qquad \alpha_2 \to \tfrac{1}{4}\gamma$$
$$\beta^{(\pm)} \to \tfrac{1}{4}\beta' \qquad \lambda^{(\pm)} \to \mp\mathrm{i}\tfrac{1}{2}f\eta \tag{6.8}$$

for $q \to 1$ where λ, γ, β' and f are q-independent parameters, we have

$$T_{12}^{(1,1)} \to \eta^{-1}\beta'\,\mathrm{e}^{+\mathrm{i}Q}x_+ \qquad T_{21}^{(0,0)} \to \gamma\,\mathrm{e}^{-\mathrm{i}Q}$$
$$T_{12}^{(0,0)} \to \beta'\,\mathrm{e}^{\mathrm{i}Q}\{fx_3^2 - x_+(P+J_3)\} \qquad T_{11}^{(1,0)} \to \eta^{-1}\lambda \tag{6.9}$$
$$T_{22}^{(0,1)} \to \lambda^{-1}\gamma\beta' x_+ \qquad T_{11}^{(0,1)} \to fx_- - \lambda(P+J_3). \tag{6.10}$$

Correspondingly,

$$\det_q T(u) \to -\tfrac{1}{4}\lambda^{-1}\gamma\beta' f(x_+x_- + x_3^2) \tag{6.11}$$

and

$$\operatorname{tr} T(u) \to \lambda u - \lambda(P+J_3) + fx_- + \lambda^{-1}\gamma\beta' x_+. \tag{6.12}$$

The corresponding rational form of $t(u)$ reads

$$t_{11}(u) = \lambda u - \lambda(P+J_3) + fx_-$$
$$t_{12}(u) = \beta'\,\mathrm{e}^{\mathrm{i}Q}\{x_+u + fx_3^2 - x_+(P+J_3)\}$$
$$t_{21}(u) = \gamma\,\mathrm{e}^{-\mathrm{i}Q} \qquad t_{22}(u) = \lambda^{-1}\gamma\beta' x_+ \tag{6.13}$$

that satisfy (2.2) for the $\check{R}$-matrix given by (2.4).

In conclusion we have found a particular solution of the truncated $T(x)$ (at $n = 4$) associated with the six-vertex form of the R-matrix. This solution is beyond those derived in terms of the Yang-Baxterization of the $T(u)$-operator [12, 13]. The realization of $T(x)$ is made through the q-deformed algebra (3.20*a*)–(3.20*f*) which include the non-commutative geometry in their own physical meaning.

Since the process of the calculations is very complicated, here we have performed the truncation only at $n = 4$. The more interesting example should be performed for large n. It deserves progress in this respect, even though it is difficult.

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Appendix

First one takes the commutator between $\hat{J}_-$ and the first line of (3.20*d*). The left-hand side is equal to

$$q^{\delta_-}(g[\hat{J}_3]_q\hat{x}_3 + \hat{J}_+[\hat{J}_-, \hat{x}_3])$$

and the right-hand side reads

$$g(q^{\delta_-} - 1)[\hat{J}_3]_q\hat{x}_3 + q^{\delta_-}\hat{J}_+[\hat{J}_-, \hat{x}_3] - [\hat{J}_-, \hat{x}_3]\hat{J}_+ - \tau_+[\hat{J}_-, K^{\delta_+}\hat{x}_+].$$

By virtue of the second line of (3.20*d*) we have

$$-[\hat{J}_-, \hat{x}_3] = (q^{\delta_+} - 1)\hat{J}_-\hat{x}_3 + \tau_- K^{-\delta_-}\hat{x}_-$$

with which and

$$K\hat{J}_\pm = q^{\mp 1}\hat{J}_\pm K$$

one obtains

$$\begin{aligned}
g(q^{\delta_-}-1)[\hat{J}_3]_q\hat{x}_3 - q^{\delta_-}\hat{J}_-\{(q^{\delta_+}-1)\hat{J}_-\hat{x}_3 + \tau_- K^{-\delta_-}\hat{x}_-\} + \{(q^{\delta_+}-1)\hat{J}_-\hat{x}_3 \\
+\tau_- K^{-\delta_-}\hat{x}_-\}\hat{J}_+ - \tau_+(\hat{J}_- K^{\delta_+}\hat{x}_+ - K^{\delta_-}\hat{x}_+\hat{J}_-) \\
= g(q^{\delta_-}-1)[\hat{J}_3]_q\hat{x}_3 + q^{\delta_-}(q^{\delta_+}-1)[\hat{J}_-,\hat{J}_+]\hat{x}_3 - \tau_- K^{-\delta_-}\hat{J}_+\hat{x}_- \\
-(q^{\delta_+}-1)\tau_+ q^{-\delta_+}K^{\delta_+}\hat{J}_-\hat{x}_+ + \tau_- K^{-\delta_-}\hat{x}_-\hat{J}_+ - \tau_+ K^{\delta_+}(q^{-\delta_+}\hat{J}_-\hat{x}_+ - \hat{x}_+\hat{J}_-) \\
= g(q^{\delta_+ +\delta_-}-1)[\hat{J}_3]_q\hat{x}_3 + \tau_-(q^{-(\delta_+ +\delta_-)}-1)K^{-\delta_-}\hat{J}_+\hat{x}_- \\
+\tau_+\{-q^{-\delta_+} - q^{-\delta_+}(q^{\delta_+}-1) + q^{-(\delta_+ +\delta_-)}\}K^{\delta_+}\hat{J}_-\hat{x}_+ + gK^{-(\delta_+ +\delta_-)}\hat{x}_3 \\
-gK^{(\delta_+ +\delta_-)}\hat{x}_3 = 0.
\end{aligned}$$

When $\delta_+ + \delta_- = 1(\mathcal{A}_+^q)$, i.e. $K - K^- = (q - q^{-1})[J_3]_q$, this leads to

$$g[\hat{J}_3]_q\hat{x}_3 + \tau_- K^{-\delta_-}\hat{J}_+\hat{x}_- + \tau_+ K^{\delta_+}\hat{J}_-\hat{x}_+ = 0$$

for $q^{-1} - 1 \neq 0$. Obviously, for $\delta_+ + \delta_- = -1(\mathcal{A}_-^q)$ we have the same result.

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